A NOTE ON HOMOTOPY TYPES OF CONNECTED COMPONENTS OF $\mathrm{Map}\,(S^4, BSU(2))$

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1. Introduction

By [Got72], connected components of $\mathrm{Map}\,(S^4, BSU(2))$ is the classifying spaces of gauge groups of principal $SU(2)$-bundles over S^4. Tsukuda [Tsu01] has investigated the homotopy types of connected components of $\mathrm{Map}\,(S^4, BSU(2))$. But unfortunately, the proof of Lemma 2.4 in [Tsu01] is not correct for $p = 2$. In this paper, we give a complete proof. Moreover, we investigate the further divisibility of ϵ_i defined in [Tsu01]. In [Tsu], it is shown that divisibility of ϵ_i have some information about A_i-equivalence types of the gauge groups.

In §2, we review the definition of ϵ_i and the motivation in homotopy theory. In §3, 4, 5 and 6, we investigate the divisibility of ϵ_i. These four sections are purely algebraic. In §7, we apply these results to A_n-types of gauge groups. Especially, we estimate the growth of the number of A_n-types of gauge groups of principal $SU(2)$-bundles over S^4.

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2. Definition and motivation

We review the definition of $\{\epsilon_i\}$. Let P_k be a principal $SU(2)$-bundle over S^4 with $\langle c_2(P_k), [S^4]\rangle = k \in \boldsymbol{Z}$. According to [Tsu], the gauge group $\mathcal{G}(P_k)$ is A_n-equivalent to a topological group $\mathcal{G}(P_0) = \mathrm{Map}\,(S^4, SU(2))$ if and only if the map

$$S^4 \vee \boldsymbol{H}P^n \xrightarrow{k\vee i} \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \xrightarrow{\nabla} \boldsymbol{H}P^\infty$$

extends over $S^4 \times \boldsymbol{H}P^n$, where $k : S^4 \to \boldsymbol{H}P^\infty$ is a classifying map of P_k, $i : \boldsymbol{H}P^n \to \boldsymbol{H}P^\infty$ is the inclusion and $\nabla : \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \to \boldsymbol{H}P^\infty$ is the folding map.

Now, we assume there exists the following homotopy commutative diagram:

$$\begin{tikzcd} S^4 \vee \boldsymbol{H}P^n \arrow[r, "k\vee i"] \arrow[d, "j"'] & \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \arrow[r, "\nabla"] & \boldsymbol{H}P^\infty \arrow[d, "\text{localization}"] \\ S^4 \times \boldsymbol{H}P^n \arrow[rr, "f"] & & \boldsymbol{H}P^\infty_{(p)} \end{tikzcd}$$

where p is a prime and $j : S^4 \vee \boldsymbol{H}P^n \to S^4 \times \boldsymbol{H}P^n$ is the inclusion.

We denote the localization of the ring of integers by the prime ideal $(p) \subset \boldsymbol{Z}$ by $\boldsymbol{Z}_{(p)}$. The p-localized complex K-theory $K(\boldsymbol{H}P^\infty_{(p)})_{(p)}$ of $\boldsymbol{H}P^\infty_{(p)}$ is computed as

$$K(\boldsymbol{H}P^\infty_{(p)})_{(p)} = \boldsymbol{Z}_{(p)}[a].$$

We may assume that there exists the generator $b \in H^4(\boldsymbol{H}P^\infty_{(p)}; \boldsymbol{Q})$ such that

$$ch\, a = \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!}.$$

Similarly, take $u \in \tilde{K}(S^4)_{(p)}$ and $s \in H^4(S^4; \boldsymbol{Q})$ such that $ch\, u = s$. Then, $f^*b = ks \times 1 + 1 \times b$ in $H^4(S^4 \times \boldsymbol{H}P^n; \boldsymbol{Q})$ and

$$f^*a = ku \times 1 + 1 \times a + \sum_{i=1}^{n} \epsilon_i(k) u \times a^i$$

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in $\tilde{K}(S^4 \times \boldsymbol{HP}^n)_{(p)}$, where $\epsilon_i(k) \in \boldsymbol{Z}_{(p)}$. We calculate $f^* ch\, a$ and $ch\, f^* a$ as follows:

$$f^* ch\, a = f^* \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!} = \sum_{j=1}^{\infty} \frac{2}{(2j)!}(ks \times 1 + 1 \times b)^j = ks \times 1 + \sum_{j=1}^{n} \left(\frac{k}{(2j+1)!} s \times b^j + \frac{2}{(2j)!} 1 \times b^j \right),$$

$$\begin{aligned} ch\, f^* a = ch \left(ku \times 1 + 1 \times a + \sum_{i=1}^{\infty} \epsilon_i(k) u \times a^i \right) &= ks \times 1 + 1 \times \sum_{j=1}^{n} \frac{2}{(2j)!} b^j + \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_i(k) s \times \left(\sum_{j=1}^{n} \frac{2}{(2j)!} b^j \right)^i \\ &= ks \times 1 + \sum_{j=1}^{n} \frac{2}{(2j)!} 1 \times b^j + \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{j_1+\cdots+j_i=l} \frac{2^i \epsilon_i(k)}{(2j_1)! \cdots (2j_i)!} s \times b^l. \end{aligned}$$

Then we have the following formula:

$$\frac{k}{(2l+1)!} = \sum_{i=1}^{l} \sum_{\substack{j_1+\cdots+j_i=l \\ j_1,\cdots,j_i \geq 1}} \frac{2^i \epsilon_i(k)}{(2j_1)! \cdots (2j_i)!}.$$

From this formula, one can see that there exists the number $\epsilon_i \in \boldsymbol{Q}$ such that $\epsilon_i(k) = \epsilon_i k$ for each i. Of course, the sequence $\{\epsilon_i\}_{i=1}^{\infty}$ satisfy the following formula for each l:

$$\frac{1}{(2l+1)!} = \sum_{i=1}^{l} \sum_{\substack{j_1+\cdots+j_i=l \\ j_1,\cdots,j_i \geq 1}} \frac{2^i \epsilon_i}{(2j_1)! \cdots (2j_i)!}.$$

For example, $\epsilon_1 = 1/6$, $\epsilon_2 = -1/180$, $\epsilon_3 = 1/1512$ etc. From the above argument, if the map (localization)$\nabla(k \vee i)$: $S^4 \vee \boldsymbol{HP}^n \to \boldsymbol{HP}^{\infty}_{(p)}$ extends over $S^4 \times \boldsymbol{HP}^n$, then $\epsilon_1 k, \cdots, \epsilon_n k \in \boldsymbol{Z}_{(p)}$.

Tsukuda [Tsu01] defines a non-negative integer $d_p(k)$ for a prime p and an integer k as the largest n such that there exists an extension of

$$S^4 \vee \boldsymbol{HP}^n \stackrel{k \vee i}{\to} \boldsymbol{HP}^{\infty} \vee \boldsymbol{HP}^{\infty} \stackrel{\nabla}{\to} \boldsymbol{HP}^{\infty} \stackrel{\text{localization}}{\to} \boldsymbol{HP}^{\infty}_{(p)}$$

over $S^4 \times \boldsymbol{HP}^n$. Remark $d_p(0) = \infty$. Clearly, if we define $\epsilon_0 = 1$, then

$$d_p(k) \leq d'_p(k) := \max\{ n \in \boldsymbol{Z}_{\geq 0} \mid \epsilon_n k \in \boldsymbol{Z}_{(p)} \}.$$

It is shown that $d_p(k) = d_p(k')$ for any prime p if the classifying spaces $B\mathcal{G}(P_k)$ and $B\mathcal{G}(P_{k'})$ are homotopy equivalent. Lemma 2.4 in [Tsu01] asserts that $d'_p(k) < \infty$ (therefore $d_p(k) < \infty$) for $k \neq 0$ and any prime p. But the proof is invalid for $p = 2$. We will give a correct proof for this case in §4.

We also review the result of [Tsu]. If $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent, then $\min\{n, d_p(k)\} = \min\{n, d_p(k')\}$ for any prime p. Let p be an odd prime. For $i < (p-1)/2$, $\epsilon_i \in \boldsymbol{Z}_{(p)}$. For $(p-1)/2 \leq i < p-1$, $p\epsilon_i \in \boldsymbol{Z}_{(p)}$. Moreover, $\epsilon_{(p-1)/2} \notin \boldsymbol{Z}_{(p)}$, $p\epsilon_{p-1} \notin \boldsymbol{Z}_{(p)}$ and $p^2 \epsilon_{p-1} \in \boldsymbol{Z}_{(p)}$. We will generalize these results in §4 and 5.

3. An explicit formula for ϵ_i

Algebraically, the sequence $\{\epsilon_i\}_{i=0}^{\infty}$ of rational numbers is defined by the following formula inductively:

$$\frac{1}{(2l+1)!} = \sum_{i=1}^{l} \sum_{\substack{j_1+\cdots+j_i=l \\ j_1,\cdots,j_i \geq 1}} \frac{2^i \epsilon_i}{(2j_1)! \cdots (2j_i)!}$$

and $\epsilon_0 = 1$. Equivalently, $\{\epsilon_i\}$ is defined by the equality

$$\sum_{l=0}^{\infty} \frac{x^l}{(2l+1)!} = \sum_{i=0}^{\infty} \epsilon_i \left(\sum_{j=1}^{\infty} \frac{2x^j}{(2j)!} \right)^i$$

in the ring of formal power series $\boldsymbol{Q}[[x]]$.

Proposition 3.1. *The rational number ϵ_i is the i-th coefficient of the Taylor expansion of $1/f'(x)$ at $0 \in \boldsymbol{C}$ for*

$$f(x) = \left(\cosh^{-1} \left(1 + \frac{x}{2} \right) \right)^2,$$

where f is holomorphic in a neighborhood of 0.

Proof. Define a holomorphic function h by

$$h(x) = 2\cosh\sqrt{x} - 2 = \sum_{i=1}^{\infty} \frac{2}{(2i)!} x^i$$

in a neighborhood of 0. Then f given by the above formula is the inverse function of h. We also define g by

$$g(x) = \sum_{i=0}^{\infty} \epsilon_i x^i.$$

Then, formally, $h'(x) = g(h(x))$ by the definition of ϵ_i. Therefore, we have $g(x) = 1/f'(x)$. □

The next proposition is proved by easy computation.

Proposition 3.2. *The holomorphic function f satisfies the following differential equation:*

$$x(x+4)f''(x) + (x+2)f'(x) - 2 = 0.$$

If the power series

$$\sum_{i=0}^{\infty} a_i x^i$$

satisfies the above equation, then

$$a_1 = 1\,,\ a_{i+1} = -\frac{i^2}{(2i+2)(2i+1)} a_i \quad (i \geq 1).$$

From these equations,

$$a_i = (-1)^{i-1} \frac{2((i-1)!)^2}{(2i)!}$$

for $i \geq 1$. Hence,

$$f(x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2((i-1)!)^2}{(2i)!} x^i$$

and

$$f'(x) = \sum_{i=0}^{\infty} (-1)^i \frac{(i!)^2}{(2i+1)!} x^i.$$

Therefore,

$$g(x) = \frac{1}{f'(x)} = \sum_{j=0}^{\infty} (-1)^j \left(\sum_{i=1}^{\infty} (-1)^i \frac{(i!)^2}{(2i+1)!} x^i \right)^j = 1 + \sum_{j=1}^{\infty} \sum_{i_1,\cdots,i_j \geq 1} (-1)^{j+i_1+\cdots+i_j} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!} x^{i_1+\cdots+i_j}.$$

This implies the following formula.

Theorem 3.3.

$$\epsilon_l = \sum_{j=1}^{l} \sum_{\substack{i_1+\cdots+i_j = l \\ i_1,\cdots,i_j \geq 1}} (-1)^{j+l} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}$$

4. Divisibility of ϵ_i by 2

For a prime p and a rational number n, we denote the p-adic valuation of n by $v_p(n)$. Equivalently, if

$$n = \frac{p^a t}{p^b s}$$

where s and t are integers prime to p, then $v_p(n) = a - b$. First, we observe the divisibility of factorials.

Lemma 4.1. *Let p be a prime. Then, for a integer*

$$n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_0$$

where $0 \leq n_i < p$ for each i,

$$v_p(n!) = \frac{1}{p-1}(n - n_0 - \cdots - n_r)$$

Proof. First, we remark the following:

$$\#\{\, k \in \mathbf{Z} \mid 1 \le k \le n, k \,\text{is divisible by}\, p^i \,\} = n_r p^{r-i} + n_{r-1} p^{r-i-1} + \cdots + n_i$$

for $1 \le i \le r$. Hence

$$\begin{aligned} v_p(n!) =& (n_r p^{r-1} + n_{r-1} p^{r-2} \cdots + n_1) + (n_r p^{r-2} + n_{r-2} p^{r-3} + \cdots + n_2) + \cdots + n_r \\ =& n_r \frac{p^r - 1}{p - 1} + n_{r-1} \frac{p^{r-1} - 1}{p - 1} + \cdots + n_1 = \frac{1}{p-1}(n - n_0 - \cdots - n_r). \end{aligned}$$

□

For $p = 2$, $v_2(n!) = n - n_0 - \cdots - n_r$.

Lemma 4.2. *For a integer*

$$n = n_r 2^r + n_{r-1} 2^{r-1} + \cdots + n_0$$

where $0 \le n_i < 2$ *for each* i*,*

$$v_2\left(\frac{(n!)^2}{(2n+1)!}\right) = -n_0 - \cdots - n_r.$$

Proof. Since $v_2((2n + 1)!) = 2n - n_0 - \cdots - n_r$ and $v_2(n!) = n - n_0 - \cdots - n_r$, the formula above follows. □

Now, we observe the divisibility of ϵ_i by 2.

Proposition 4.3. *For positive integers* $n_1, \cdots, n_m$ *and* $l = n_1 + \cdots + n_m$*, if* $n_j \ge 2$ *for some* j*, then*

$$2^{l-1} \frac{(n_1!)^2 \cdots (n_m!)^2}{(2n_1 + 1)! \cdots (2n_m + 1)!} \in \mathbf{Z}_{(2)}.$$

Proof. From Lemma 4.2,

$$2^n \frac{(n!)^2}{(2n+1)!} \in \mathbf{Z}_{(2)}.$$

Moreover, if $n > 1$,

$$2^{n-1} \frac{(n!)^2}{(2n+1)!} \in \mathbf{Z}_{(2)}.$$

The conclusion follows form this. □

From this proposition and Theorem 3.3,

$$\epsilon_l \equiv 6^{-l} \mod 2^{-l+1}\mathbf{Z}_{(2)}.$$

Then we have the following theorem.

Theorem 4.4.

$$v_2(\epsilon_l) = -l$$

Then $d_2'(k) = v_2(k)$ (see §2 for the definition of d_2') and Lemma 2.4 in [Tsu01] for $p = 2$ is proved.

5. Divisibility of ϵ_i by an odd prime

In general, divisibility of ϵ_i by an odd prime p is more complicated than by 2 because the interval between a multiple of p and the next one is longer. But for $p = 3$, we will have a similar result.

Lemma 5.1. *Let* p *be a prime. Then, for a integer*

$$n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_0$$

where $0 \le n_i < p$ *for each* i*,*

$$v_p((2n + 1)!) \le \frac{2}{p-1}(n - n_0 - \cdots - n_r) + r + 1$$

Proof. First, we remark the following:

$$\#\{\, k \in \mathbf{Z} \mid 1 \le k \le 2n+1, k \text{ is divisible by } p^i \,\} \le 2(n_r p^{r-i} + n_{r-1} p^{r-i-1} + \cdots + n_i) + 1$$

for $1 \le i \le r+1$. Hence

$$\begin{aligned} v_p((2n+1)!) \le& 2(n_r p^{r-1} + n_{r-1} p^{r-2} \cdots + n_1) + 1 + 2(n_r p^{r-2} + n_{r-2} p^{r-3} + \cdots + n_2) + 1 + \cdots + 2n_r + 1 + 1 \\ =& \frac{2}{p-1}(n - n_0 - \cdots - n_r) + r + 1. \end{aligned}$$

□

Lemma 5.2. *For an odd prime p and a positive integer n,*

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \ge -\frac{2n}{p-1}.$$

Moreover, this equality holds if and only if $n = (p-1)/2$.

Proof. Let $n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_0$ where $0 \le n_i < p$ for each i, especially $n_r \neq 0$. From Lemma 4.1 and 5.1,

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \ge -r - 1 > -\frac{2n}{p-1}$$

for $n \ge p$ since

$$\frac{p-1}{2}(r+1) < p^r \le n.$$

For $n < p$,

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) = \begin{cases} 0 & (0 \le n < (p-1)/2) \\ -1 & ((p-1)/2 \le n < p) \end{cases}.$$

Then

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \ge -\frac{2n}{p-1}$$

holds for any n and the equality holds if and only if $n = (p-1)/2$. □

This lemma implies the next proposition.

Proposition 5.3. *For an odd prime p, positive integers $n_1, \cdots, n_m$ and $l = n_1 + \cdots + n_m$, then*

$$v_p\left(\frac{(n_1!)^2 \cdots (n_m!)^2}{(2n_1+1)! \cdots (2n_m+1)!}\right) \ge -\frac{2l}{p-1},$$

where the equality holds if and only if $n_i = (p-1)/2$ for each i.

Then, by Theorem 3.3, we have

$$\epsilon_{n(p-1)/2} \equiv (-1)^{n(p+1)/2} \frac{\left(\frac{p-1}{2}!\right)^{2n}}{(p!)^n} \mod p^{-n+1}\mathbf{Z}_{(p)}.$$

Theorem 5.4. *For a non-negative integer n,*

$$v_p(\epsilon_{n(p-1)/2}) = -n.$$

Especially, $v_3(\epsilon_l) = -l$. We also have the following estimate.

Theorem 5.5. *For a non-negative integer $l < n(p-1)/2$,*

$$v_p(\epsilon_l) > -n$$

Proof. Let positive integers $i_1, \cdots, i_m$ satisfy $i_1 + \cdots + i_m = l$. From Proposition 5.3,

$$v_p\left(\frac{(i_1!)^2 \cdots (i_m!)^2}{(2i_1+1)! \cdots (2i_m+1)!}\right) \ge -\frac{2l}{p-1} > -n,$$

Therefore, by Theorem 3.3, $v_p(\epsilon_l) > -n$. □

These results imply $d'_p(k) = (p-1)v_p(k)/2$.

6. Further observation

Though it suffices to know Theorem 4.2, 5.4 and 5.5 for our application, we see the divisibility by 5 here.

For $l = 2n$, by Theorem 5.4, $v_5(\epsilon_{2n}) = -n$. Then we consider the case $l = 2n+1$. Since $v_5(\epsilon_{2n+1}) \geq -n$,

$$\epsilon_{2n+1} \equiv (-1)^{n+1} n \cdot \frac{(2!)^{2n-2}(3!)^2}{(5!)^{n-1}7!} + (-1)^n(n+1) \cdot \frac{(2!)^{2n}(1!)^2}{(5!)^n 3!} \mod 5^{-n+1}\boldsymbol{Z}_{(5)}.$$

The right hand side is computed as

$$(-1)^n \frac{2^{2n}(7-2n)}{(5!)^{n-1}7!}.$$

Then if $l \equiv 3 \mod 10$, $v_5(\epsilon_l) > -[l/2]$, where $[l/2]$ represents the largest integer $\leq l/2$. On the other hand, if $l \not\equiv 3 \mod 10$, $v_5(\epsilon_l) = -[l/2]$.

Actually, ϵ_l ($l = 1, \cdots, 20$) is computed as follows, where the right hand sides are the prime factorizations.

$$\begin{aligned}
\epsilon_1 &= 2^{-1}3^{-1}\\
\epsilon_2 &= 2^{-2}3^{-2}5^{-1}(-1)\\
\epsilon_3 &= 2^{-3}3^{-3}5^{0}7^{-1}\\
\epsilon_4 &= 2^{-4}3^{-4}5^{-2}7^{-1}(-1)23\\
\epsilon_5 &= 2^{-5}3^{-5}5^{-2}7^{-1}11^{-1}263\\
\epsilon_6 &= 2^{-6}3^{-6}5^{-3}7^{-2}11^{-1}13^{-1}(-1)353 \cdot 379\\
\epsilon_7 &= 2^{-7}3^{-7}5^{-3}7^{-2}11^{-1}13^{-1}197 \cdot 797\\
\epsilon_8 &= 2^{-8}3^{-8}5^{-4}7^{-2}11^{-1}13^{-1}17^{-1}(-1)383 \cdot 42337\\
\epsilon_9 &= 2^{-9}3^{-9}5^{-4}7^{-3}11^{-1}13^{-1}17^{-1}19^{-1}2689453969\\
\epsilon_{10} &= 2^{-10}3^{-10}5^{-5}7^{-2}11^{-2}13^{-1}17^{-1}19^{-1}(-1)26893118531\\
\epsilon_{11} &= 2^{-11}3^{-11}5^{-5}7^{-3}11^{-2}13^{-1}17^{-1}19^{-1}23^{-1}73 \cdot 76722629153\\
\epsilon_{12} &= 2^{-12}3^{-12}5^{-6}7^{-4}11^{-2}13^{-2}17^{-1}19^{-1}23^{-1}(-1)127 \cdot 563 \cdot 46721395729\\
\epsilon_{13} &= 2^{-13}3^{-13}5^{-5}7^{-4}11^{-2}13^{-2}17^{-1}19^{-1}23^{-1}71 \cdot 1531 \cdot 20479 \cdot 397849\\
\epsilon_{14} &= 2^{-14}3^{-14}5^{-7}7^{-4}11^{-2}13^{-2}17^{-1}19^{-1}23^{-1}29^{-1}(-1)43 \cdot 19981442744694143\\
\epsilon_{15} &= 2^{-15}3^{-15}5^{-7}7^{-5}11^{-3}13^{-2}17^{-1}19^{-1}23^{-1}29^{-1}31^{-1}233 \cdot 11874127314767975461\\
\epsilon_{16} &= 2^{-16}3^{-16}5^{-8}7^{-5}11^{-3}13^{-2}17^{-2}19^{-1}23^{-1}29^{-1}31^{-1}(-1)319473088311274492668499\\
\epsilon_{17} &= 2^{-17}3^{-17}5^{-8}7^{-5}11^{-3}13^{-2}17^{-2}19^{-1}23^{-1}29^{-1}31^{-1}103 \cdot 191 \cdot 11677 \cdot 8295097 \cdot 229156549\\
\epsilon_{18} &= 2^{-18}3^{-18}5^{-9}7^{-6}11^{-3}13^{-3}17^{-2}19^{-2}23^{-1}29^{-1}31^{-1}37^{-1}(-1)811 \cdot 236696258753425486925956793\\
\epsilon_{19} &= 2^{-19}3^{-19}5^{-9}7^{-6}11^{-3}13^{-3}17^{-2}19^{-2}23^{-1}29^{-1}31^{-1}37^{-1}276162497983 \cdot 959905866507242503\\
\epsilon_{20} &= 2^{-20}3^{-20}5^{-10}7^{-6}11^{-4}13^{-3}17^{-2}19^{-2}23^{-1}29^{-1}31^{-1}37^{-1}41^{-1}(-1)269 \cdot 13677071637569 \cdot 225347651134721497
\end{aligned}$$

7. Applications to A_n-types of gauge groups

As in §2, we assume there exists the following homotopy commutative diagram:

$$\begin{tikzcd}
S^4 \vee \boldsymbol{H}P^n \arrow[r, "k \vee i"] \arrow[d, "j"'] & \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \arrow[r, "\nabla"] & \boldsymbol{H}P^\infty \arrow[d, "\text{localization}"] \\
S^4 \times \boldsymbol{H}P^n \arrow[rr, "f"] & & \boldsymbol{H}P^\infty_{(p)}
\end{tikzcd}$$

where p is a prime and i and j are the inclusions. Let us consider the map

$$S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \xrightarrow{f \cup ((\text{localization})i)} \boldsymbol{H}P^\infty_{(p)}.$$

The obstruction to extending this map over $S^4 \times \boldsymbol{H}P^{n+1}$ lives in $\pi_{4n+7}(\boldsymbol{H}P^{\infty}_{(p)})$. Then, from Theorem of [Sel78], the obstruction to extending the map

$$S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \overset{(p\times\mathrm{id})\cup\mathrm{id}}{\to} S^4 \times \boldsymbol{H}P^n \cup * \times \boldsymbol{H}P^{n+1} \overset{f\cup((\mathrm{localization})i)}{\to} \boldsymbol{H}P^{\infty}_{(p)}.$$

over $S^4 \times \boldsymbol{H}P^{n+1}$ vanishes for an odd prime p. Hence one can see $d_p(pk) > d_p(k)$ and $d_p(k) \geq v_p(k)$ inductively. For $p = 2$, from [Jam57], $d_2(4k) > d_2(k)$ and $d_2(k) \geq [v_2(k)/2]$ similarly. Then we have

$$v_p(k) \leq d_p(k) \leq \frac{p-1}{2} v_p(k)$$

for an odd prime p and

$$\left[\frac{v_2(k)}{2}\right] \leq d_2(k) \leq v_2(k)$$

from previous two sections. Especially, $d_3(k) = v_3(k)$.

Now we give the lower bound of the number of A_n-types of gauge groups of principal $SU(2)$-bundle over S^4. As stated in §2, if $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent, then $\min\{n, d_p(k)\} = \min\{n, d_p(k')\}$ for any prime p. If p is an odd prime, then

$$\#\{\min\{n, d_p(k)\} \mid k \in \boldsymbol{Z}\} \geq \left[\frac{2n}{p-1} + 1\right]$$

since $0 = d_p(1) < d_p(p) < d_p(p^2) < \cdots < d_p(p^{[2n/(p-1)]}) \leq n$. If $p = 2$, then

$$\#\{\min\{n, d_2(k)\} \mid k \in \boldsymbol{Z}\} \geq \left[\frac{n}{2} + 1\right]$$

since $0 = d_2(1) < d_2(4) < d_2(16) < \cdots < d_2(4^{[n/2]}) \leq n$.

Theorem 7.1. *The number of A_n-types of gauge groups of principal $SU(2)$-bundles over S^4 is greater than*

$$\left[\frac{n}{2} + 1\right] \prod_{p:\text{odd prime}} \left[\frac{2n}{p-1} + 1\right].$$

We can express the logarithm of this as follows:

$$\begin{aligned}
\log\left(\left[\frac{n}{2} + 1\right] \prod_{p:\text{odd prime}} \left[\frac{2n}{p-1} + 1\right]\right) &= \log\left[\frac{n}{2} + 1\right] + \sum_{p:\text{odd prime}} \log\left[\frac{2n}{p-1} + 1\right] \\
&= \log\left[\frac{n}{2} + 1\right] + \sum_{r=2}^{n+1} \left(\pi\left(\frac{2n}{r-1} + 1\right) - 1\right)(\log r - \log(r-1)) \\
&= \log\left[\frac{n}{2} + 1\right] + \sum_{r=1}^{n} \pi\left(\frac{2n}{r} + 1\right) \log\left(1 + \frac{1}{r}\right) - \log(n+1),
\end{aligned}$$

where π is the prime counting function. The second equality is seen by

$$\#\left\{p : \text{an odd prime} \,\middle|\, \left[\frac{2n}{p-1} + 1\right] \geq r\right\} = \pi\left(\frac{2n}{r-1} + 1\right) - 1.$$

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